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1.061 / 1.61 Transport Processes in the Environment
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6. Continuous point sources



In contrast to previous sections, this chapter describes the continuous release of chemicals into a fluid (e.g. smoke released from stacks, groundwater flux into a river, industrial waste streams etc.). A chemical released continuously into a moving fluid will form a plume that grows in the lateral dimension through diffusion and extends downstream via advection. This chapter describes the concentration field of the plume in both bounded and unbounded domains. In a bounded domain (e.g. a narrow, shallow channel), the steady-state solution is obtained by a simple mass balance; the transient period before steady-state is far more interesting, and the analytical expression describing the concentration front is derived here. The animation highlights how the observed concentration front varies with distance from the source.

The sample problems test the user's ability to simplify problems based on the geometry of the domain, and also to obtain information on the diffusion coefficient based on the observed concentration front.

6. Continuous Release - Point Source

A scalar released continuously into a moving fluid and from a discrete point will form a plume that grows in the lateral dimension through diffusion and extends downstream via advection. Because the concentration profile perpendicular to the flow is established by Fickian diffusion from the centerline, it has a Gaussian distribution. For this reason the plume is called a Gaussian Plume. Some common examples that approximate a Gaussian plume include smoke released from a stack, a waste stream released into river, and the contaminant plume emerging from a leaking drum buried underground.

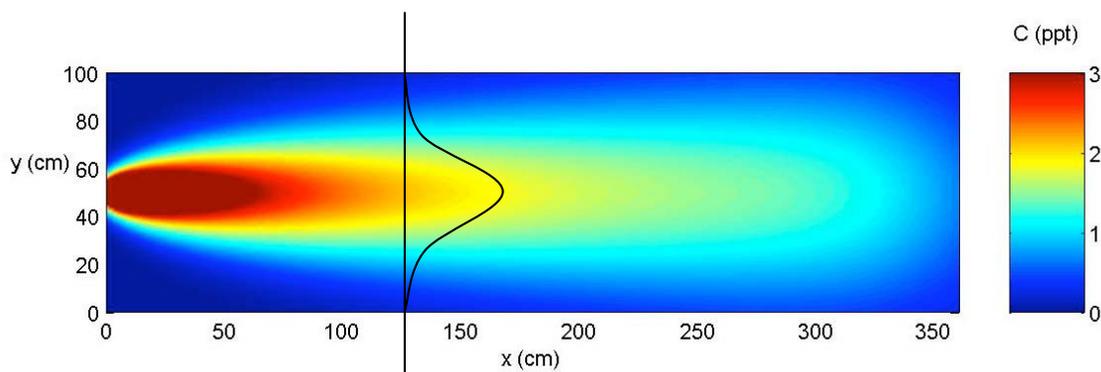


Figure 1. Concentration contours for a Gaussian plume originating at $x=0$, $y=50$ cm, $z=0$. The concentration profiles $C(y)$ and $C(z)$ are Gaussian. For example, see the profile $C(x=125, y, z=0)$ shown above in black.

A release may be considered continuous if the advection time scale is short compared to the duration of the release. And a release approximates a point source if the scale of the source is small compared to the distance from the source and the width of the plume at that distance. Whether a source approximates a continuous or instantaneous release, and whether it approximates a point or distributed source depends not only on the configuration of the source, but also on the spatial position of interest. Thus, the choice of approximations does not depend on the absolute spatial and temporal scales of the source, but rather on the relative scales. For example, consider a river with mean flow U . The concentration observed during and after a ten-minute release (T_{release}) of chemical into this river will approach the predictions for a continuous release for downstream locations $x \ll UT_{\text{release}}$ but will approach an instantaneous release at $x \gg UT_{\text{release}}$.

Steady-State, One-Dimensional Solution for a Continuous Release

Consider a long, narrow channel of width L_y , depth L_z , unidirectional flow U , and isotropic diffusion, D . At mid-width and mid-depth in this channel we release a continuous stream of tracer at a rate, \dot{m} [kg s^{-1}]. Neglecting reaction, the Conservation of Mass equation (transport equation) is,

$$(1) \quad \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} = D \left[\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right].$$

Typically a river is advection dominated such that $Pe \gg 1$ for any distance of interest downstream ($Pe = xU/D \gg 1$ for any x of interest). For this condition, the longitudinal diffusion term will be negligible when compared to the longitudinal advection term, *i.e.* $D\partial^2C/\partial x^2 \ll u\partial C/\partial x$, and the former term will be dropped. We assume that the release has been occurring sufficiently long that the conditions in the river have reached steady-state ($\partial C/\partial t = 0$). Specifically, in an advected dominated system and at a distance x downstream of the release the steady-state will be reached at time $t \gg x/U$. Furthermore, we assume that the time-scale to mix the tracer uniformly over the width and depth, T_{mix} , is sufficiently short that we may assume the tracer is instantly uniform over the cross-section. Then $\partial C/\partial y = \partial C/\partial z = 0$. This is satisfied at distances $x \gg UT_{mix}$. With these additional assumptions, (1) reduces to

$$(2) \quad u \frac{\partial C}{\partial x} = 0,$$

or simply, $C \neq f(x)$ in the far field of a continuous, bounded release. To find the far-field concentration value we enforce mass balance within a control volume around the release.

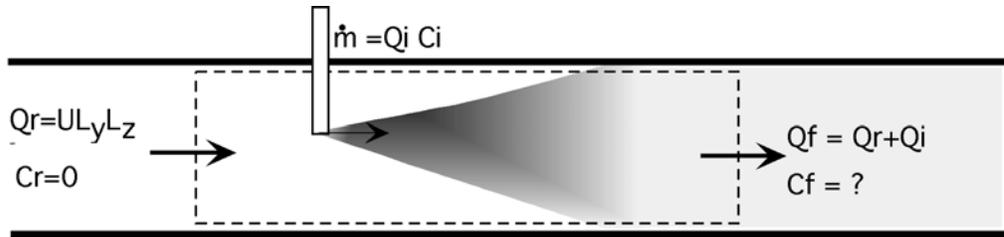


Figure 2. Mass Balance within a Control Volume around a Steady, Continuous Source. Top view of the channel with flow Q_r and continuous injection of concentration C_i at rate Q_i .

In steady-state, the total mass in the control volume is not a function of time, and the sum of influxes and outfluxes of mass must balance, *i.e.*,

$$Q_r C_r + Q_i C_i = Q_f C_f = (Q_r + Q_i) C_f$$

influxes outflux

With $C_r = 0$, we find the far-field, steady concentration.

$$(3) \quad C_f = (Q_i C_i) / (Q_i + Q_r).$$

Dilution Gauging:

The steady, far-field solution is sometimes used to estimate the volumetric flow in rivers with shallow complex bathymetry, for which the deployment of velocity meters is impractical. The tracer release is designed so that the injection flow rate, Q_i , is far less than the stream flow, Q_r , so that $(Q_r + Q_i \approx Q_r)$. The injection concentration and injection rate are known, and the far-field concentration, C_f , is measured. Then from (3),

$$(4) \quad Q_r = Q_i C_i / C_f.$$

Recall that (4) requires the tracer to be well-mixed over the channel cross-section at the point that C_f is measured. The concentration of a conservative tracer (no sources or sinks) will cease to be a function of downstream distance once it is well-mixed over the cross-section. So, to establish the position at which (4) can be applied, measure the concentration progressively downstream until it asymptotes to a constant value.

Steady, Continuous, Point-Source in an Unbounded Domain

Next consider a steady, continuous, point source in an unbounded domain. Without boundaries a well-mixed condition can never be reached, so all dimensions must be retained. We assume that $v = w = 0$, neglect reaction; and use $Pe \gg 1$ to neglect longitudinal diffusion relative to advection. For generality we allow anisotropy in diffusion [$D_x \neq D_y \neq D_z$]. The equation governing steady transport in this system is,

$$(5) \quad u \frac{\partial C}{\partial x} = D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2}.$$

The above equation indicates that advection and diffusion are decoupled by coordinate. That is, transport in the x-direction is by advection only and transport in the y- and z-directions is by diffusion only. To solve (5) we recast it in a frame of reference moving with the flow. Let $\tau = x/u$, then $u \partial C / \partial x = \partial C / d(x/u) = \partial C / \partial \tau$, and (5) becomes

$$(6) \quad \frac{\partial C}{\partial \tau} = D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2}.$$

Equation (6) describes the evolution of tracer within a thin slab of fluid moving downstream at speed u . The slab has length dx . As the slab passes the injection point, it receives a slug of mass $M = \dot{m} (dx/u)$, where dx/u represents the duration of time the slab is exposed to the injection. If the injection is located at $x = 0$, the slab receives the slug of mass at $\tau = 0$. Thus, the initial condition within the slab is $C(\tau = 0) = M \delta(y) \delta(z)$, for an injection located at $y = z = 0$. The solution to (6) with this initial condition was given in Chapter 3, equation (23). Modifying that solution to the coordinates used here,

$$(7) \quad C(y, z, \tau) = \frac{M}{dx 4\pi \tau \sqrt{D_y D_z}} \exp\left(-\frac{y^2}{4D_y \tau} - \frac{z^2}{4D_z \tau}\right).$$

We return (7) to the stationary frame by substituting $\tau = x/u$. Additionally, $M = \dot{m} (dx/u)$. This gives us the concentration field for a steady, unbounded, 3-D, continuous release (\dot{m}) from point source at $(x, y, z) = 0$

$$(8) \quad C(x, y, z) = \frac{\dot{m}}{4\pi \sqrt{D_y D_z x}} \exp\left(-\frac{uy^2}{4D_y x} - \frac{uz^2}{4D_z x}\right)$$

Downstream Evolution of Continuous Plume in a Channel

Now we consider the downstream evolution of a continuous source (Fig. 3). The release is at mid-depth ($z = 0$), mid-width ($y = 0$) and $x = 0$. To account for the boundary conditions at the channel walls and the water surface we must add image sources to the unbounded solution given in (8). Recall from Chapter 4 that a no-flux boundary can be satisfied with a positive image source and a totally absorbing boundary can be satisfied with a negative image source. If the chemical in the plume is not volatile, the water surface acts as a no-flux boundary. The four no-flux boundaries are satisfied with a sum of positive images located at $(x=0, y=nL_y, z=0)$ and $(x=0, y=0, z=nL_z)$ with $n = \pm 1, \pm 2, \pm 3$ upward to \pm infinity.

(9)

$$C(x,y,z) = \frac{\dot{m}}{4\pi\sqrt{D_y D_z} x} \left[\sum_{n=-\infty}^{\infty} \exp\left(-\frac{u(y+nL_y)^2}{4D_y x} - \frac{uz^2}{4D_z x}\right) + \sum_{\substack{n=-\infty \\ \text{but } n \neq 0}}^{\infty} \exp\left(-\frac{uy^2}{4D_y x} - \frac{u(z+nL_z)^2}{4D_z x}\right) \right]$$

In the second sum $n = 0$ is excluded as it represents the real source which is already accounted for in the first sum.

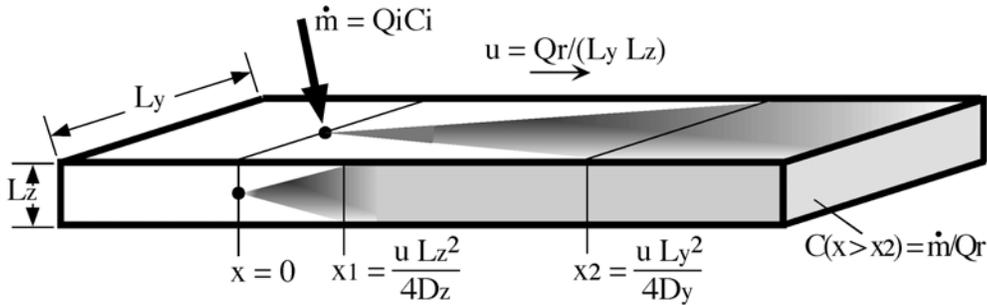


Figure 3. Continuous Release at mid-width ($y=0$), mid-depth ($z = 0$) and $x = 0$. The points at which the plume becomes uniformly mixed over depth (x_1) and width (x_2) are indicated.

The plume concentration will be well-mixed over depth at $x > u L_z^2/4D_z$, and $\partial C/\partial z = 0$ beyond this distance (Figure 3). This removes the vertical diffusion term from (5). Following the same mathematical development following (5) for a three-dimensional plume, an unbounded two-dimensional plume released from $(x,y)=0$ has the concentration field.

$$(10) \quad C(x,y) = \frac{\dot{m}/u}{L_z \sqrt{4\pi D_y (x/u)}} \exp\left(-\frac{uy^2}{4D_y x}\right)$$

With $\partial C/\partial z = 0$, the no-flux boundaries at the surface and bed are automatically satisfied. We only need to account for the side boundaries with image sources at $(x=0, y=nL_y, z=0)$ for $n = \pm 1, \pm 2$, upward to \pm infinity. The concentration in the plume between x_1 and x_2 , as shown in Figure 3, is then approximately

$$(11) \quad C(x, y) = \frac{\dot{m}/u}{L_z \sqrt{4\pi D_y (x/u)}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{u(y + nL_y)^2}{4D_y x}\right),$$

where again, $n = 0$ represents the real source.

Transient, One-Dimensional Solution for Continuous, Point Source in a Channel

Figure 3 represents the spatial evolution downstream of a continuous source after steady-state has been reached. Now we consider the transient period between when the injection starts ($t = 0$) and the concentration field reaches a steady state ($t \gg x/U$). Specifically, we allow $\partial C/\partial t \neq 0$. For simplicity we assume a well-mixed condition over width and depth, such that $\partial C/\partial y = \partial C/\partial z = 0$, and we assume no reaction.

$$(12) \quad \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = D_x \frac{\partial^2 C}{\partial x^2}$$

We retain the longitudinal diffusion term, even though we expect $Pe \gg 1$. In Figure 4, below, we see that the diffusion term creates spatial detail around the front, or leading edge of the plume. Once the injection is initiated, each slab of fluid, of width dx , passing the release point receives mass $M = \dot{m} (dx/u)$. We assume that the mass instantly mixes within the slab, *i.e.* over the channel depth and width and over the length dx , such that the initial concentration in the slab is $C(x=0) = C_0 = M/(L_y L_z dx) = \dot{m}/(u L_y L_z)$. The boundary conditions then become,

$$(13) \quad \begin{aligned} C(x) &= 0 & \forall x, t < 0 \\ C_0 &= \frac{\dot{m}}{u L_y L_z} & x = 0, t \geq 0 \end{aligned}$$

The solution to (12) with boundary conditions in (13) is

One-Dimensional, Transient Solution to Continuous Release

$$(14) \quad C(x, t) = \frac{\dot{m}}{2 u L_y L_z} \operatorname{erfc}\left(\frac{x - ut}{\sqrt{4D_x t}}\right) = \frac{C_0}{2} \operatorname{erfc}\left(\frac{x - ut}{\sqrt{4D_x t}}\right).$$

The function, *erfc*, is called the complementary error function, and is defined in Table 1 below. The distribution in (14) is called a *Break-Through Curve*, because it represents the temporal evolution of concentration when tracer first *breaks through* at a specific distance downstream of an injection. Figure 4 provides an example for a tracer injected

at $x = 0$ beginning at $t = 0$ into a channel with current speed u . If the tracer mixes quickly over depth, L_z , and width, L_y , then (14) predicts the concentration measured at $x = L_x$,

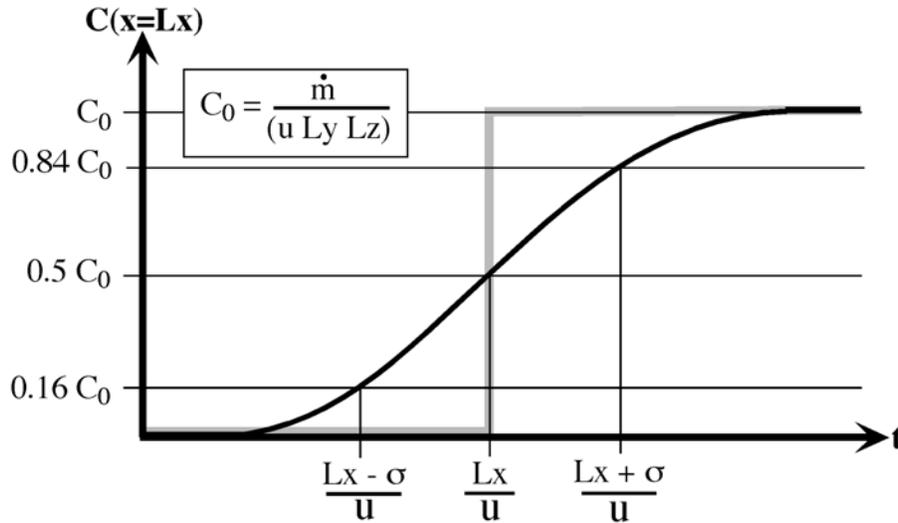


Figure 4. Break-Through Curve. Injection of tracer at rate \dot{m} is initiated at $t = 0$. The concentration observed at $x = L_x$, shown as a black curve, follows (14). The break-through curve shown in gray results if diffusion is neglected, $D_x = 0$.

Table 1. The error function, $erf(a)$, is the integral of the Gaussian distribution from zero to a . The complementary error function is the integral of the Gaussian distribution from a to infinity, or simply, $erfc(a) = 1 - erf(a)$. In addition, $erfc(-a) = 2 - erfc(a)$.

$$erf(a) = \frac{2}{\sqrt{\pi}} \int_0^a e^{-r^2} dr \quad erfc(a) = \frac{2}{\sqrt{\pi}} \int_a^{\infty} e^{-r^2} dr$$

a	erf(a)	erfc(a)	a	erf(a)	erfc(a)
0	0	1.0	1.4	0.952285	0.047715
0.7	0.677801	0.322199	2.0	0.995322	0.004678
1.0	0.842701	0.157299	3.0	0.999978	0.000022

Example: Break-Through Curves with Gaussian Plume.

You can open the animation on the Chapter 6 homepage. The animation depicts the evolution of the concentration field downstream of a point source located at $(x = 0, y = 50 \text{ cm}, z = 0)$ and initiated at $t = 0$ in a channel with steady flow $u=1 \text{ cm/s}$ and longitudinal diffusion coefficient $D = 2 \text{ cm}^2/\text{s}$. The concentration is measured at three points all at elevation $z = 0$. At each point the center of the front, defined by $C = 0.5 C_{\text{final}}$, arrives at the advection time scale, x/u . The duration of the front, which is the time required for the concentration to rise from $C=0$ to C_{final} , is $4\sigma_i/u$, where σ_i is the length-scale of the front at $t = x/u$, *i.e.* $\sigma_i = \sqrt{2D_x x/u}$. Use the table below to compare the transport time-scales with the concentration record at each Probe.

Probe	x[cm]	y[cm]	x/u [s]	σ_i [cm]	$4\sigma_i/u$ [s]
A	100	50	100	20	80
B	250	50	250	32	128
C	250	0	250	32	128

